(1)

(1.3)

2×2 Matrices

Def: A 2×2 matrix is an array

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where a_{11}, a_{12}, a_{21}, and a_{22} are four scalars.

For example $\begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 13 & 7 \\ 6 & 11 \end{pmatrix}$

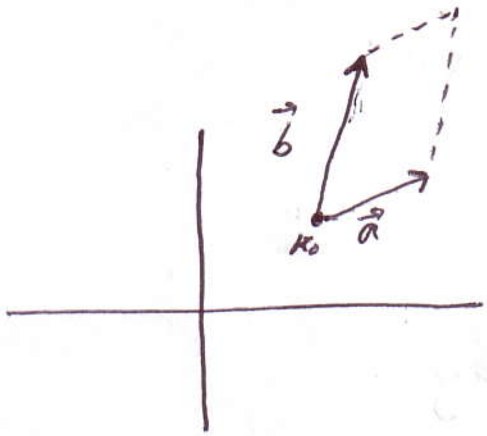
are 2×2 matrices.

The indices ij in a_{ij} represent the row and the column in which the scalar is found.

For example, 0 in $\begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$ is in the second row and in the first column. hence $0 = a_{21}$. what are the indices of 4?

Determinants of 2×2 Matrices

Recal that $A = \{K_0 + s\vec{a} + t\vec{b} ; s,t \in [0,1]\}$ is the set of points in a parallelogram with corner K_0, spanned by $\vec{a}$ and $\vec{b}$.

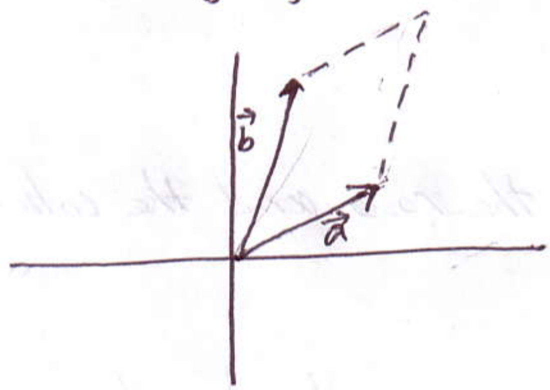


$\vec{a} = (a_1, a_2)$

$\vec{b} = (b_1, b_2)$

Naturally we might ask what is the area of this parallelogram, $V(A)$?

We can translate the parallelogram to the origin without changing its area:

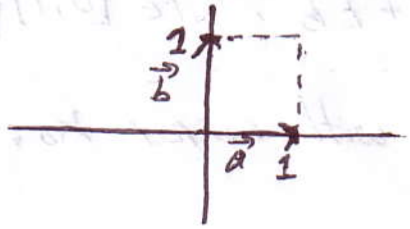


The area depends on $\vec{a}$ and $\vec{b}$ but it <u>does not</u> depend on x_0!

Lets say $V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = V\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix} \equiv$ Area of parallelogram spanned by $\vec{a}$ and $\vec{b}$.

First, lets examine a few simple cases:

If $\vec{a} = (1,0)$ and $\vec{b} = (0,1)$, what is $V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = V\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$?

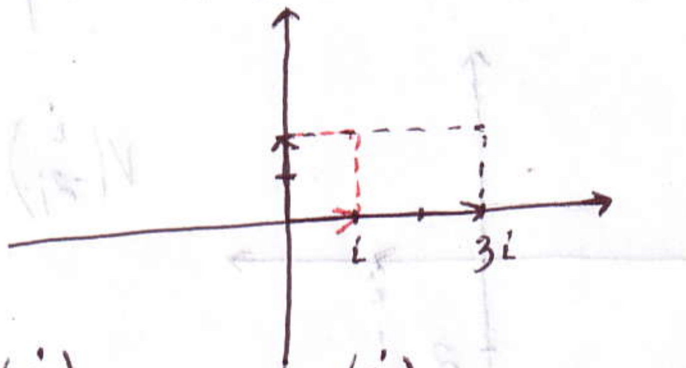


$V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = 1$

Note that $\vec{a}$ is really $\vec{i}$ and $\vec{b}$ is $\vec{j}$. Hence, we see that

$V\begin{pmatrix}\vec{i}\\\vec{j}\end{pmatrix} = 1.$

what if $\vec{a} = 3i$ and $\vec{b} = 2j$?

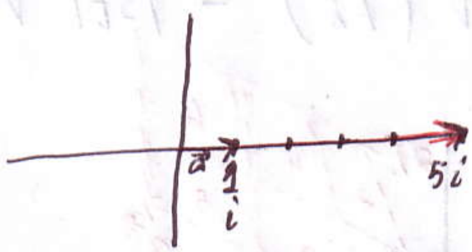


$$V\begin{pmatrix}3i\\2j\end{pmatrix} = 3V\begin{pmatrix}i\\2j\end{pmatrix} = 3\cdot 2\,V\begin{pmatrix}i\\j\end{pmatrix} = 3\cdot 2\cdot 1$$

what if $\vec{a} = si$ $\vec{b} = tj$, $s,t > 0$

$$V\begin{pmatrix}si\\tj\end{pmatrix} = st\,V\begin{pmatrix}i\\j\end{pmatrix} = st\cdot 1. = st$$

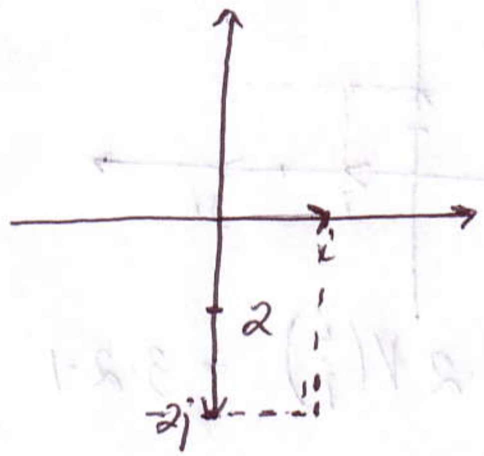
what would you say about $V\begin{pmatrix}i\\5i\end{pmatrix}$?



$$V\begin{pmatrix}i\\5i\end{pmatrix} = 5V\begin{pmatrix}i\\i\end{pmatrix} = 0 \quad (\text{Why?})$$

It seems that if $\vec{a}$ and $\vec{b}$ are linearly dependent, that is, if $\vec{a} = k\vec{b}$ for some $k \in \mathbb{R}$ or if either $\vec{a}$ or $\vec{b}$ is $\vec{0}$ $V\binom{\vec{a}}{\vec{b}} = 0$. (Why?)

what happens if $\vec{a} = i$ and $\vec{b} = -2j$?



$$V\binom{i}{-2j} = 2 = |-2|\,V\binom{i}{j}$$

More generally, if $t, s \in \mathbb{R}$ are <u>any</u> scalars, what is

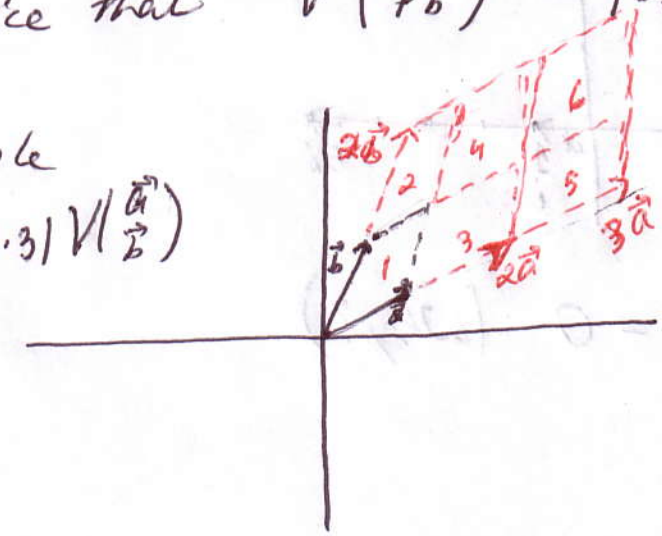
$V\binom{si}{tj}$? Answer: $|st|\,V\binom{i}{j}$.

Finally, notice that $V\binom{s\vec{a}}{t\vec{b}} = |st|\,V\binom{\vec{a}}{\vec{b}} = |st\,V\binom{\vec{a}}{\vec{b}}|$

For example

$V\binom{3\vec{a}}{2\vec{b}} = |2\cdot 3|\,V\binom{\vec{a}}{\vec{b}}$

(why?)



Observe that if $\vec{a} = a_1 i + a_2 j$ and $\vec{b} = b_1 i + b_2 j$

then $V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = V\begin{pmatrix}a_1 i + a_2 j\\ \vec{b}\end{pmatrix} = V\begin{pmatrix}\vec{a}\\ b_1 i + b_2 j\end{pmatrix} =$

$V\begin{pmatrix}a_1 i + a_2 j\\ b_1 i + b_2 j\end{pmatrix}$

It is therefore natural to ask whether there exists a simple relationship between $V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}$ and $V\begin{pmatrix}a_1 i\\ \vec{b}\end{pmatrix}$, $V\begin{pmatrix}a_2 j\\ \vec{b}\end{pmatrix}$

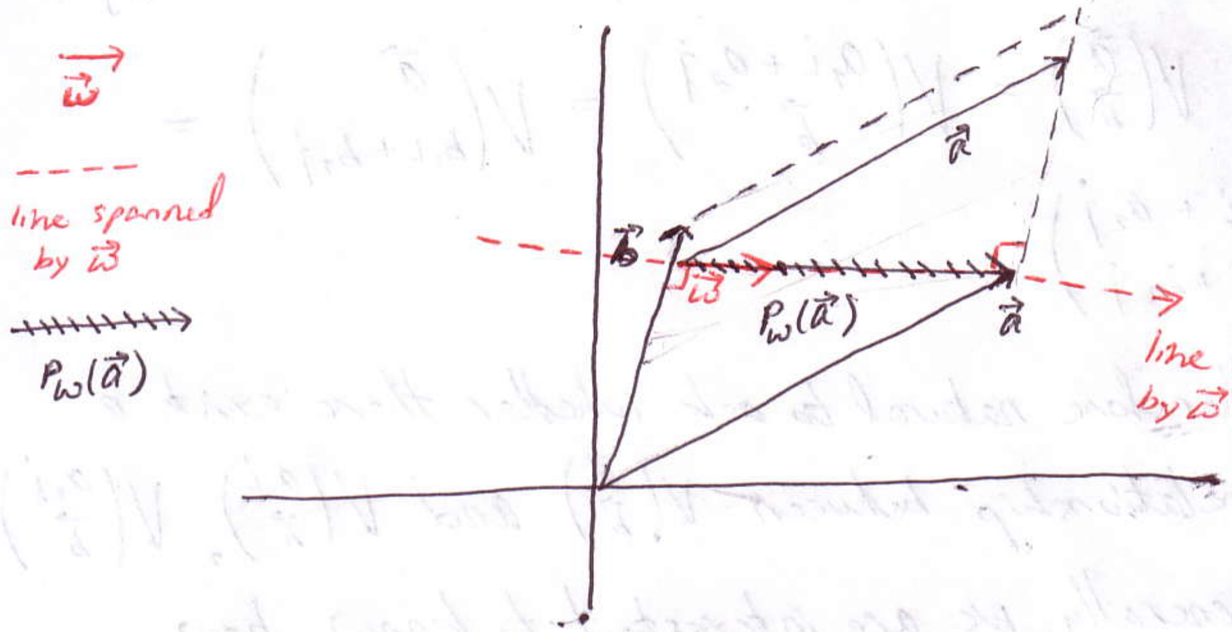
More generally, we are interested to know how $V\begin{pmatrix}\vec{a}+\vec{c}\\ \vec{b}\end{pmatrix}$ is determined by $V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}$ and $V\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}$.

To answer this question, notice that the area of a parallelogram with base b and height h is bh.

Thus, two parallelograms with the same base and height have the same area:

In our case this means that if $\vec{\omega}$ is orthogonal to $\vec{b}$, then

$V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = V\begin{pmatrix}P_\omega(\vec{a})\\ \vec{b}\end{pmatrix}$ where $P_\omega(\vec{a})$ is the orthogonal projection of $\vec{a}$ onto ω.



By the same reasoning, if $\vec{w}'$ is orthogonal to $\vec{a}$ then

$V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = V\begin{pmatrix}\vec{a}\\ P_{\vec{w}'}(b)\end{pmatrix}$ (Make sure that you can verify it!)

This is good news because $V\begin{pmatrix}\vec{a}+\vec{c}\\ \vec{b}\end{pmatrix} = V\begin{pmatrix}P_\omega(\vec{a}+\vec{c})\\ \vec{b}\end{pmatrix} =$

$= V\begin{pmatrix}\left[\frac{(\vec{a}+\vec{c})\cdot\vec{w}}{\|\omega\|^2}\right]\vec{w}\\ \vec{b}\end{pmatrix}$. But $\frac{(\vec{a}+\vec{c})\cdot\vec{\omega}}{\|\omega\|^2}$ is a scalar, so from

previous work we know that $V\begin{pmatrix}\frac{(\vec{a}+\vec{c})\cdot\vec{\omega}}{\|\omega\|^2}\vec{\omega}\\ \vec{b}\end{pmatrix} =$

$$= \left|\frac{(\vec{a}+\vec{c})\cdot\vec{w}}{\|\vec{\omega}\|^2}\right| V\begin{pmatrix}\vec{\omega}\\ \vec{b}\end{pmatrix} = \left|\frac{\vec{a}\cdot\vec{w}}{\|\vec{w}\|^2} + \frac{\vec{c}\cdot\vec{w}}{\|\vec{w}\|^2}\right| V\begin{pmatrix}\vec{w}\\ \vec{b}\end{pmatrix} =$$

$$= \left|\frac{\vec{a}\cdot\vec{\omega}}{\|\vec{\omega}\|^2} V\begin{pmatrix}\vec{w}\\ \vec{b}\end{pmatrix} + \frac{\vec{c}\cdot\vec{\omega}}{\|\vec{\omega}\|^2} V\begin{pmatrix}\vec{w}\\ \vec{b}\end{pmatrix}\right| = \left|V\begin{pmatrix}\frac{\vec{a}\cdot\vec{\omega}}{\|\vec{w}\|^2}\vec{\omega}\\ \vec{b}\end{pmatrix} \pm V\begin{pmatrix}\frac{\vec{c}\cdot\vec{w}}{\|\omega\|^2}\vec{\omega}\\ \vec{b}\end{pmatrix}\right|$$

$$= \left|V\begin{pmatrix}P_\omega(\vec{a})\\ \vec{b}\end{pmatrix} \pm V\begin{pmatrix}P_\omega(\vec{c})\\ \vec{b}\end{pmatrix}\right| = \left|V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} \pm V\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}\right|$$

Notice that the last equality has a "±" in it. This is because $\frac{\vec{a}\cdot\vec{w}}{\|\vec{w}\|^2} V\begin{pmatrix}\vec{w}\\ \vec{b}\end{pmatrix} = \pm\left|\frac{\vec{a}\cdot\vec{w}}{\|\vec{w}\|^2}\right| V\begin{pmatrix}\vec{w}\\ \vec{b}\end{pmatrix} =$

$= \pm V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}$ and similarly $\frac{\vec{c}\cdot\vec{w}}{\|\vec{w}\|^2} V\begin{pmatrix}\vec{w}\\ \vec{b}\end{pmatrix} = \pm V\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}$

The quantity $\left|\pm V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} \pm V\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}\right|$ is equivalent to $\left|V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} \pm V\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}\right|$

We have shown that $V\begin{pmatrix}\vec{a}+\vec{c}\\ \vec{b}\end{pmatrix} = \left|V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} \pm V\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}\right|$.

The same argument demonstrates that $V\begin{pmatrix}\vec{a}\\ \vec{b}+\vec{d}\end{pmatrix} = \left|V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} \pm V\begin{pmatrix}\vec{a}\\ \vec{d}\end{pmatrix}\right|$

Let's summarize what we know so far:

1) If $\vec{a}$ and $\vec{b}$ are linearly dependent $V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = 0$

2) $V\begin{pmatrix}t\vec{a}+s\vec{c}\\ \vec{b}\end{pmatrix} = \left|tV\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} \pm sV\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}\right|$ for $s, t \in \mathbb{R}$

3) In particular, if $s = 0$ $V\begin{pmatrix}t\vec{a}\\ \vec{b}\end{pmatrix} = \left|tV\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}\right|$

Hence $V\begin{pmatrix}\vec{a}+s\vec{c}\\ \vec{b}\end{pmatrix}$ is 'almost' $\left|V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} + sV\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}\right|$

If we manage to define a function $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ s.t. $f\begin{pmatrix}s\vec{a}\\ t\vec{b}\end{pmatrix} = st\, f\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}$ and $f\begin{pmatrix}s\vec{a}+t\vec{c}\\ \vec{b}\end{pmatrix} = sf\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} + tf\begin{pmatrix}\vec{c}\\ \vec{b}\end{pmatrix}$,

$f\begin{pmatrix}\vec{a}\\ s\vec{b}+t\vec{c}\end{pmatrix} = sf\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} + tf\begin{pmatrix}\vec{a}\\ \vec{c}\end{pmatrix}$, then it will follow

$V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = \left|f\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}\right|$

$$f\begin{pmatrix}\vec{a}\\\vec{b}\end{pmatrix} = f\begin{pmatrix}a_1 i + a_2 j\\ b_1 i + b_2 j\end{pmatrix} = a_1 f\begin{pmatrix}i\\ b_1 i + b_2 j\end{pmatrix} + a_2 f\begin{pmatrix}j\\ b_1 i + b_2 j\end{pmatrix}$$

$$= a_1\left(b_1 f\begin{pmatrix}i\\i\end{pmatrix} + b_2 f\begin{pmatrix}i\\j\end{pmatrix}\right) + a_2\left(b_1 f\begin{pmatrix}j\\i\end{pmatrix} + b_2 f\begin{pmatrix}j\\j\end{pmatrix}\right)$$

$$= a_1 b_2 f\begin{pmatrix}i\\j\end{pmatrix} + a_2 b_1 f\begin{pmatrix}j\\i\end{pmatrix} \quad \text{where } f\begin{pmatrix}i\\i\end{pmatrix} = f\begin{pmatrix}j\\j\end{pmatrix} = 0$$

(why?)

To simplify this further, notice that $0 = f\begin{pmatrix}i+j\\i+j\end{pmatrix} =$

$$= f\begin{pmatrix}i\\i\end{pmatrix} + f\begin{pmatrix}i\\j\end{pmatrix} + f\begin{pmatrix}j\\i\end{pmatrix} + f\begin{pmatrix}j\\j\end{pmatrix} = f\begin{pmatrix}i\\j\end{pmatrix} + f\begin{pmatrix}j\\i\end{pmatrix}$$

Thus $f\begin{pmatrix}j\\i\end{pmatrix} = -f\begin{pmatrix}i\\j\end{pmatrix}$

Hence $a_1 b_2 f\begin{pmatrix}i\\j\end{pmatrix} + a_2 b_1 f\begin{pmatrix}j\\i\end{pmatrix} = a_1 b_2 f\begin{pmatrix}i\\j\end{pmatrix} - a_2 b_1 f\begin{pmatrix}i\\j\end{pmatrix} =$

$$= (a_1 b_2 - a_2 b_1) f\begin{pmatrix}i\\j\end{pmatrix} = a_1 b_2 - a_2 b_1 \quad \text{where } f\begin{pmatrix}i\\j\end{pmatrix} = f\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = 1$$

We have arrived at the following definition:

Def: Given $\vec{a}, \vec{b}$, $\det\begin{pmatrix}\vec{a}\\\vec{b}\end{pmatrix} = \det\begin{pmatrix}a_1 & a_2\\b_1 & b_2\end{pmatrix} = a_1 b_2 - a_2 b_1$

Furthermore, the area of a parallelogram spanned by $\vec{a}, \vec{b}$ is given by $V\begin{pmatrix}\vec{a}\\\vec{b}\end{pmatrix} = \left|\det\begin{pmatrix}\vec{a}\\\vec{b}\end{pmatrix}\right|$.

Sometimes the word 'det' is ommitted, and the determinant is represented by straight lines $||$: $\begin{vmatrix}a_1 & a_2\\b_1 & b_2\end{vmatrix} = a_1 b_2 - a_2 b_1$

Ex. Find the value of the determinant

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1-1 = 0 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2$$

What about $\begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix}$? (Hint: if $\vec{a} = (1,2)$ and $\vec{b} = (3,4)$

$$\begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix} = \det\left(\begin{pmatrix} \vec{a} \\ -2\vec{b} \end{pmatrix}\right).$$

If $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -3$ what is $\det\begin{pmatrix} a_{11}-3a_{21} & a_{12}-3a_{22} \\ 2a_{21} & 2a_{22} \end{pmatrix}$.

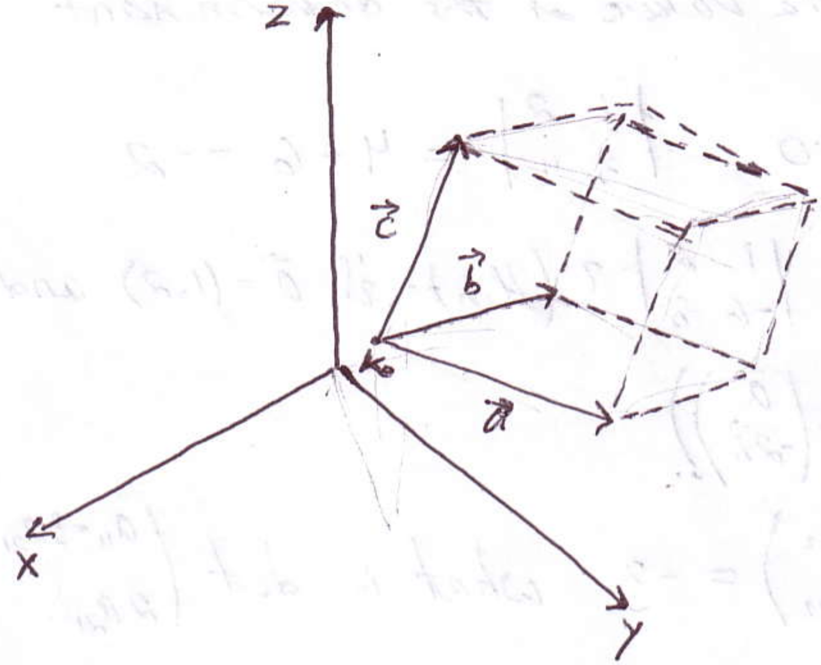
3×3 Matrices and determinants

Def: A 3×3 matrix is an array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where each a_{ij} is a scalar

To motivate 3×3 determinants observe that, given vectors $\vec{a}, \vec{b}, \vec{c}$, the set $A = \{k_0 + r\vec{a} + s\vec{b} + t\vec{c}\}$ defines a 3-D shape, called parallelopiped provided that $r, s, t \in [0,1]$ and $k_0 \in \mathbb{R}^3$.



A parallelopiped is a box all of whose faces are parallelograms. We are interested in finding its volume $V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix}$.

Similar considerations to those which have lead our intuition in the 2×2 case suggest that $V\begin{pmatrix}r\vec{a}\\ s\vec{b}\\ t\vec{c}\end{pmatrix} = \left|rst\,V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix}\right|$

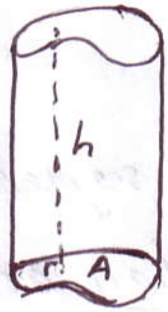
for any scalars r, s, t (you should try to justify this property)

It might also hold true that $V\begin{pmatrix}\vec{a}\\ \vec{b}+s\vec{d}\\ \vec{c}\end{pmatrix} = \left|V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} \pm s\,V\begin{pmatrix}\vec{a}\\ \vec{d}\\ \vec{c}\end{pmatrix}\right|$,

$V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}+s\vec{d}\end{pmatrix} = \left|V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} \pm s\,V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{d}\end{pmatrix}\right|$, and $V\begin{pmatrix}\vec{a}+s\vec{d}\\ \vec{b}\\ \vec{c}\end{pmatrix} =$

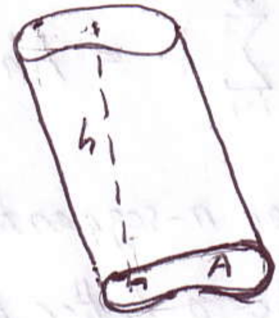
$= \left|V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} \pm s\,V\begin{pmatrix}\vec{d}\\ \vec{b}\\ \vec{c}\end{pmatrix}\right|$

To see that this is indeed the case, observe that the volume of a cylinder with height h and base of area A is given by hA:



$V = Ah$

(a)



$V = Ah$

(b)

While (a) is obvious (why?), some explanation is necessary to justify (b):

$h_1 + h_2 + h_3 = h$

$h_1 = h_2 = h_3 = \frac{h}{3}$

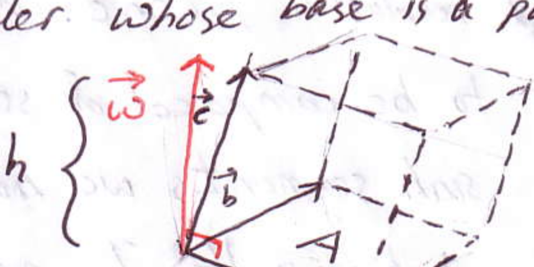
(c)

We can think of (b) as a 'heigh resolution' picture. Under a lower resolution, (b) will appear to be composed of segments similar in form to (a). The more such segments we have, the more miniscule is the difference between the 'low resolution' and the 'heigh resolution' pictures. Consequently, the volume of the 'low resolution' shape is a good approximation to the

'heigh resolution' shape. For instance, $Vol_b \approx Vol_c$ where b is the 'heigh resolution' shape and c is the segmented figure in fig. (c). In other words, $Vol_b \approx Vol_c =$
$= \sum_{i=1}^{3} A h_i = A \sum_{i=1}^{3} h_i = Ah$.

More generally, an n-segmented shape with each segment having height h/n will be a better approximation to b than an n-1-segmented shape with each segment having height $\frac{h}{n-1}$.
However, $Vol_{c_n} = \sum_{i=1}^{n} A h_i = \sum_{i=1}^{n} A \frac{h}{n} = Ah \sum_{i=1}^{n} \frac{1}{n} =$
$= Ah \, n \cdot \frac{1}{n} = Ah$ for all n (c_n is the n-segmented approximation to b. Note that (a) $= c_1$.)
It follows that $Vol_{c_1} = Vol_{c_n} = Ah \Rightarrow Vol_b = Vol_a =$
$= Vol_c$

In our particular case, observe that a parallelopiped is just a cylinder whose base is a parallelogram.



$h = \| P_\omega(\vec{c}) \|$
$\vec{\omega} \perp$ to plane through $\vec{a}$ and $\vec{b}$.

Hence, the volume of a parallelopiped is the area of one of its faces (a parallelogram!) multiplied by the height.

In other words, $V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = V\begin{pmatrix}\vec{a}\\ \vec{b}\\ P_{\vec{\omega}}(\vec{c})\end{pmatrix}$ where $\vec{\omega}$ is orthogonal to the plane through $\vec{a}$ and $\vec{b}$.

This implies that $V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}+t\vec{d}\end{pmatrix} = \left| V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} \pm t V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{d}\end{pmatrix}\right|$ (why?)

Similar result holds for $V\begin{pmatrix}\vec{a}\\ \vec{b}+t\vec{d}\\ \vec{c}\end{pmatrix}$ and $V\begin{pmatrix}\vec{a}+t\vec{d}\\ \vec{b}\\ \vec{c}\end{pmatrix}$ (be able to establish this!).

Finally, observe that for a 'parallelopiped' spanned (generated) by $\vec{a}_1\ \vec{a}_2\ \vec{a}_3$ $V\begin{pmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{pmatrix} = 0$ if $\vec{a}_i = \vec{a}_j$ for $i \neq j$

(can you justify this?)

Hence, we want to find a function $f: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longmapsto \mathbb{R}$ such that $V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = \left| f\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix}\right|$ where $f\begin{pmatrix}\vec{a}\\ \vec{b}+t\vec{d}\\ \vec{c}\end{pmatrix} =$

$= f\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} + t f\begin{pmatrix}\vec{a}\\ \vec{d}\\ \vec{c}\end{pmatrix}$ for any $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{R}^3$, $t \in \mathbb{R}$, and

$f\begin{pmatrix}\vec{a}\\ \vec{a}\\ \vec{c}\end{pmatrix} = 0$ (Note that similar results hold for, say $f\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}+t\vec{d}\end{pmatrix}$, and $f\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{b}\end{pmatrix}$).

To derive an explicit formula for $f\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = f\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}$,

Observe that $f\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = f\begin{pmatrix}a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\\ b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\\ c_1\vec{i} + c_2\vec{j} + c_3\vec{k}\end{pmatrix} =$

$$= f\begin{pmatrix}a_1\vec{i}\\ b_2\vec{j}\\ c_3\vec{k}\end{pmatrix} + f\begin{pmatrix}a_1\vec{i}\\ b_3\vec{k}\\ c_2\vec{j}\end{pmatrix} + f\begin{pmatrix}a_2\vec{j}\\ b_1\vec{i}\\ c_3\vec{k}\end{pmatrix} + f\begin{pmatrix}a_2\vec{j}\\ b_3\vec{k}\\ c_1\vec{i}\end{pmatrix}$$

$$+ f\begin{pmatrix}a_3\vec{k}\\ b_1\vec{i}\\ c_2\vec{j}\end{pmatrix} + f\begin{pmatrix}a_3\vec{k}\\ b_2\vec{j}\\ c_1\vec{i}\end{pmatrix} = a_1b_2c_3\, f\begin{pmatrix}i\\ j\\ k\end{pmatrix} + a_1b_3c_2\, f\begin{pmatrix}i\\ k\\ j\end{pmatrix}$$

$$+ a_2b_1c_3\, f\begin{pmatrix}j\\ i\\ k\end{pmatrix} + a_2b_3c_1\, f\begin{pmatrix}j\\ k\\ i\end{pmatrix} + a_3b_1c_2\, f\begin{pmatrix}k\\ i\\ j\end{pmatrix} + a_3b_2c_1\, f\begin{pmatrix}k\\ j\\ i\end{pmatrix}.$$

To simplify this further observe that interchanging any two vectors in $f\begin{pmatrix}i\\ j\\ k\end{pmatrix}$, say $f\begin{pmatrix}j\\ i\\ k\end{pmatrix}$, the result is the same as multiplying $f\begin{pmatrix}i\\ j\\ k\end{pmatrix}$ by (-1). In other words, $f\begin{pmatrix}j\\ i\\ k\end{pmatrix} = -f\begin{pmatrix}i\\ j\\ k\end{pmatrix}$. We will prove this shortly, but for now, lets apply this idea to the problem at hand:

$f\begin{pmatrix}i\\ j\\ k\end{pmatrix} = 1$ (why?), $f\begin{pmatrix}k\\ j\\ i\end{pmatrix} = -f\begin{pmatrix}j\\ k\\ i\end{pmatrix} = (-1)(-1)f\begin{pmatrix}j\\ i\\ k\end{pmatrix} =$

$= (-1)(-1)(-1)f\begin{pmatrix}i\\ j\\ k\end{pmatrix} = (-1)^3(1) = -1$

Similarly, $f\begin{pmatrix}i\\ k\\ j\end{pmatrix} = -1$, $f\begin{pmatrix}j\\ i\\ k\end{pmatrix} = -1$, $f\begin{pmatrix}k\\ i\\ j\end{pmatrix} = 1$, $f\begin{pmatrix}j\\ k\\ i\end{pmatrix} = 1$

It follows that $f\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = f\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{pmatrix} =$

$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$

The function f is important enough to have a name:

Def: let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$

Then $\det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 +$
$+ a_3b_1c_2 - a_3b_2c_1$. Furthermore $V\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = \left|\det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix}\right|$.

How are we to remember this formula?

Observe that $\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{pmatrix} = a_1(b_2c_3 - b_3c_2) -$

$- a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = a_1\det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3\end{pmatrix} - a_2\det\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3\end{pmatrix}$

$+ a_3\det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2\end{pmatrix}$ (1)

Without some mnemonic device, formula (1) would be difficult to memorize. The rule to learn is that you move along the first row, multiplying a_{1j} by the determinant of the 2×2 matrix obtained by canceling out the first row and the j^{th} column, and then you add these up, remembering to put a minus in front of a_2. For example, the determinant multiplied by the middle term of formula (1), namely $\det\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3\end{pmatrix}$ is obtained by crossing out the first row and the second column of the given 3×3 matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$. Insted of the prefix "det", determinants are sometimes signified by $|*|$.

Ex. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 1.$

(why is this result obvious?)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3+12-9=0$$

Properties of Determinants

Property 1: Interchanging any two rows in the determinant results in a change of sign.

Proof: Without loss of generality, we will show that interchanging $\vec{b}, \vec{c}$ in $\det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$ will result in a change of sign.

Observe that $\det\begin{pmatrix} \vec{a} \\ \vec{b}+\vec{c} \\ \vec{c}+\vec{b} \end{pmatrix} = \pm V\begin{pmatrix} \vec{a} \\ \vec{b}+\vec{c} \\ \vec{c}+\vec{b} \end{pmatrix} = 0$ (why?)

Hence $0 = \det\begin{pmatrix} \vec{a} \\ \vec{b}+\vec{c} \\ \vec{c}+\vec{b} \end{pmatrix} = \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c}+\vec{b} \end{pmatrix} + \det\begin{pmatrix} \vec{a} \\ \vec{c} \\ \vec{c}+\vec{b} \end{pmatrix} =$

$= \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} + \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{b} \end{pmatrix} + \det\begin{pmatrix} \vec{a} \\ \vec{c} \\ \vec{c} \end{pmatrix} + \det\begin{pmatrix} \vec{a} \\ \vec{c} \\ \vec{b} \end{pmatrix}$

$$= \det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} + \det\begin{pmatrix}\vec{a}\\ \vec{c}\\ \vec{b}\end{pmatrix} \quad \left(\det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{b}\end{pmatrix} = \det\begin{pmatrix}\vec{a}\\ \vec{c}\\ \vec{c}\end{pmatrix} = 0 \text{ (why?)}\right)$$

Hence $0 = \det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} + \det\begin{pmatrix}\vec{a}\\ \vec{c}\\ \vec{b}\end{pmatrix}$ from which

$-\det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = \det\begin{pmatrix}\vec{a}\\ \vec{c}\\ \vec{b}\end{pmatrix}$ and the result follows.

Property 2: Multiplying any row by a scalar $\alpha \in \mathbb{R}$ results in multiplication of the entire determinant by α.

Proof: This was already established by the motivation behind det.

Property 3: If we change a row by adding another row to it, the value of the determinant remains the same.

Proof: Without loss of generality, it suffices to show that

$\det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = \det\begin{pmatrix}\vec{a}\\ \vec{b}+\vec{a}\\ \vec{c}\end{pmatrix}$.

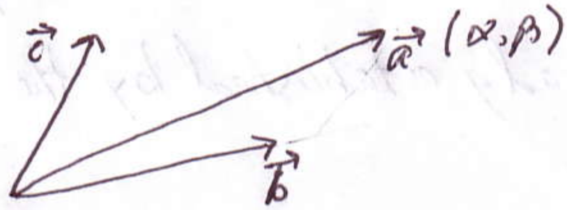
We know however that $\det\begin{pmatrix}\vec{a}\\ \vec{b}+\vec{a}\\ \vec{c}\end{pmatrix} = \det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} + \det\begin{pmatrix}\vec{a}\\ \vec{a}\\ \vec{c}\end{pmatrix}$

$= \det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} + 0 = \det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix}$ Thus the result follows.

Ex. Suppose $\vec{a} = \alpha \vec{b} + \beta \vec{c}$ where $(a_1, a_2, a_3) = \vec{a} =$
$= \alpha(b_1, b_2, b_3) + \beta(c_1, c_2, c_3)$

Compute $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$

Solution: Geometrically, $\vec{a} = \alpha\vec{b} + \beta\vec{c}$ means that $\vec{a}$ lies in the plane spanned by $\vec{b}$ and $\vec{c}$:



In other words, if we treat $\vec{b}, \vec{c}$ as generators of a coordinate system, then the coordinates of $\vec{a}$ are (α, β).

It follows that $V\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = 0$ (because $r\vec{a} + s\vec{b} + t\vec{c} : r, s, t \in \mathbb{R}$ is a deformed parallelopiped that has no volume; this 'parallelopiped' is actually only a parallelogram!)

Hence $0 = V\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \left|\det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}\right| \Rightarrow \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = 0.$

Alternatively, $\det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \det\begin{pmatrix} \alpha\vec{b} + \beta\vec{c} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \det\begin{pmatrix} \alpha\vec{b} \\ \vec{b} \\ \vec{c} \end{pmatrix} +$

$+ \det\begin{pmatrix} \beta\vec{c} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \alpha \det\begin{pmatrix} \vec{b} \\ \vec{b} \\ \vec{c} \end{pmatrix} + \beta \det\begin{pmatrix} \vec{c} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \alpha \cdot 0 + \beta \cdot 0 = 0$

The cross (outer) product

Given vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ we want to obtain a vector $\vec{x} = (x, y, z)$ that is perpendicular to both $\vec{a}$ and $\vec{b}$. The vector $\vec{x}$ must be nonzero and satisfy $\vec{x} \cdot \vec{a}$ and $\vec{x} \cdot \vec{b} = 0$:

$$a_1 x + a_2 y + a_3 z = 0 \quad (1)$$

$$b_1 x + b_2 y + b_3 z = 0 \quad (2)$$

To accomplish this, recall that for any $\vec{c} = (c_1, c_2, c_3)$

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot (c_1, c_2, c_3)$$

$$= \left(\vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot \left(c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k} \right)$$

Notice that if we pretend that $\vec{i}, \vec{j}, \vec{k}$ are scalars, the above expression can be written symbolically as

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1, c_2, c_3) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Observe now that

$$0 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (a_1, a_2, a_3)$$

where (c_1, c_2, c_3) was replaced by (a_1, a_2, a_3)

Also

$$0 = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (b_1, b_2, b_3)$$

where (c_1, c_2, c_3) was replaced by (b_1, b_2, b_3)

In particular, the vector $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is perpendicular

to both $\vec{a}$ and $\vec{b}$ and so we can set $\vec{x}$ to be this vector.

This motivates the following definition.

Def: The cross product of vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is the vector given by $\vec{a} \times \vec{b} =$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex. Find $(3\vec{i}-\vec{j}+\vec{k})\times(\vec{i}+2\vec{j}-\vec{k})$

Solution:

$$(3\vec{i}-\vec{j}+\vec{k})\times(\vec{i}+2\vec{j}-\vec{k}) = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

Algebraic properties of the cross product

(i) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$ In particular $\vec{a}\times\vec{b}$ is not commutative.

(ii) $\vec{a}\times(\beta\vec{b}+\gamma\vec{c}) = \beta(\vec{a}\times\vec{b})+\gamma(\vec{a}\times\vec{c})$ and $(\alpha\vec{a}+\beta\vec{b})\times\vec{c} =$
$= \alpha(\vec{a}\times\vec{c})+\beta(\vec{b}\times\vec{c})$ (prove properties (i) & (ii)!)

Note that $\vec{a}\times\vec{a} = -(\vec{a}\times\vec{a})$ by property (i). Thus $\vec{a}\times\vec{a} = 0$

In particular

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$$

Also, $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$

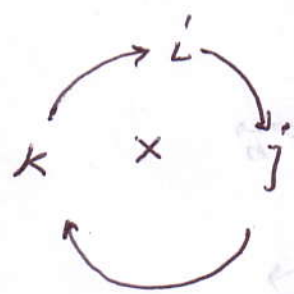


Diagram that helps remembering this.

Next, we calculate the length of $\vec{a}\times\vec{b}$. Note that

$$\|\vec{a}\times\vec{b}\|^2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 =$$

$$= (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2$$

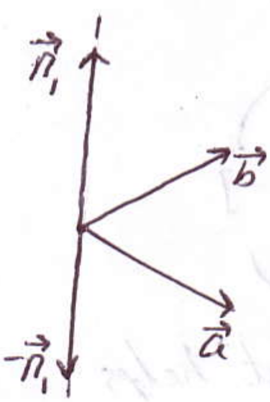
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$ which equals

$\|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})\cos^2\theta = \|\vec{a}\|\|b\|^2\sin^2\theta$ where θ satisfies $0 \le \theta \le \pi$. Upon taking square root, we get

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\,|\sin\theta| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$$

Finally note that, since $\vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0$ it follows that for $\vec{c} = \alpha\vec{a} + \beta\vec{b}$ $\vec{c}\cdot(\vec{a}\times\vec{b}) = 0$ (why?) In other words, $\vec{a}\times\vec{b}$ is perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$, with length $\|\vec{a}\|\,\|\vec{b}\|\sin\theta = \|\vec{a}\times\vec{b}\|$.

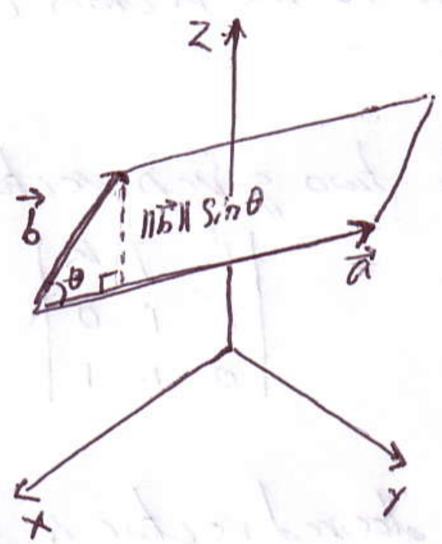
There are still two possible vectors that satisfy these conditions:



$\vec{a}\times\vec{b} = \vec{n}_1$ This is known as the right-hand rule!

Area of parallelogram in $\mathbb{R}^3$

We have seen that 2×2 determinants can describe the area of a parallelogram in $\mathbb{R}^2$. Similarly, the volume of parallelopipeds in $\mathbb{R}^3$ can be computed with 3×3 determinants. How might we compute the surface area of a parallelopiped? This question is easily answered if we can figure out the area of a parallelogram in $\mathbb{R}^3$.



The area of a parallelogram is base times height.
If θ is the angle between $\vec{a}$ and $\vec{b}$, then the height h is given by $\|\vec{b}\| \sin\theta$. The base, $\|\vec{a}\|$, times the height $\|\vec{b}\| \sin\theta$ is therefore the area of our parallelogram.
However, by the previous work, $\|\vec{a}\|\|\vec{b}\|\sin\theta = \|\vec{a}\times\vec{b}\|$
It follows that the area of a parallelogram spanned by vectors $\vec{a}$ and $\vec{b}$ is equal to $\|\vec{a}\times\vec{b}\|$.

Ex, Find the area of the parallelogram spanned by the two vectors $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = -\vec{i} - \vec{k}$.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 0 & -1 \end{vmatrix} = -2\vec{i} - 2\vec{j} + 2\vec{k}$$

Thus the area is $\|\vec{a} \times \vec{b}\| = \sqrt{(-2)^2 + (-2)^2 + 2^2} = 2\sqrt{3}$.

Ex. Find a unit vector orthogonal to the vectors $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$.

Solution:

A vector perpendicular to the two given vectors may be obtained by $(\vec{i} + \vec{j}) \times (\vec{j} + \vec{k}) = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{i} - \vec{j} + \vec{k}$

Because $\|\vec{i} - \vec{j} + \vec{k}\| = \sqrt{3}$ the desired vector is $\frac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{k})$

Remark: the vector $\frac{1}{\sqrt{3}}(-\vec{i} + \vec{j} - \vec{k})$ is also a solution (why?)

Ex, Derive an identity relating the dot and cross products from the formulas

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta \quad \text{and} \quad \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

by eliminating θ.

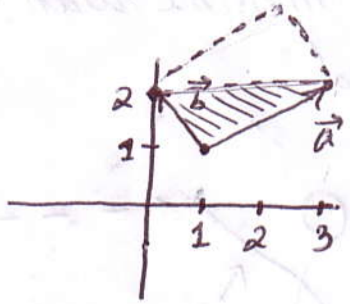
Solution: Notice that $\left(\|\vec{u}\| \|\vec{v}\| \sin\theta\right)^2 + \left(\|\vec{u}\| \|\vec{v}\| \cos\theta\right)^2 = \left(\|u\| \|v\|\right)^2$

From this it follows that $\|\vec{u} \times \vec{v}\|^2 + (\vec{u} \cdot \vec{v})^2 = \|\vec{u}\|^2 \|\vec{v}\|^2$

Hence $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

Ex. Find the area of the triangle with vertices at the points $(1,1)$, $(0,2)$ and $(3,2)$

Solution:



The area of a triangle spanned by $\vec{a}$ and $\vec{b}$ is half of the area of the corresponding parallelogram.
Since we are in $\mathbb{R}^2$ it suffices to compute $\det\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}$
where $\vec{b} = (0,2)-(1,1) = (-1,1)$ and $\vec{a} = (3,2)-(1,1) = (2,1)$

Hence $\det\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 2-(-1) = 3 = |3| = V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}$

It follows that the area of a parallelogram spanned by $\vec{a}$ and $\vec{b}$ is 3. The area of our triangle is $\frac{1}{2}V\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} = \frac{3}{2}$.

Ex. Find the volume of the parallelopiped spanned by the three vectors $\vec{i}+3\vec{k}$, $2\vec{i}+\vec{j}-2\vec{k}$, and $5\vec{i}+4\vec{k}$.

let $\vec{a} = (1,0,3)$ $\vec{b} = (2,1,-2)$ $\vec{c} = (5,0,4)$

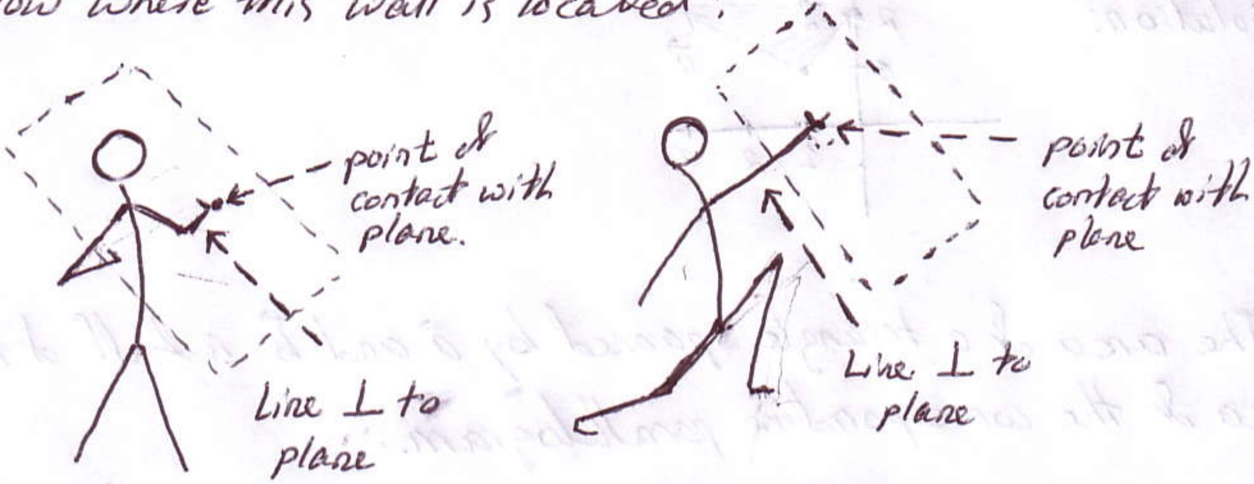
then $\det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 5 & 0 & 4 \end{vmatrix} =$

$= \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix} = 4 + 3(-5) = 4 - 15 = -11$

So the volume is 11.

Equations of Planes

Consider a mime touching some invisible wall. How do you know where this wall is located?



If you can see a line segment perpendicular to the plane as well as the point where this line segment meets the plane, you can deduce the location of this plane.

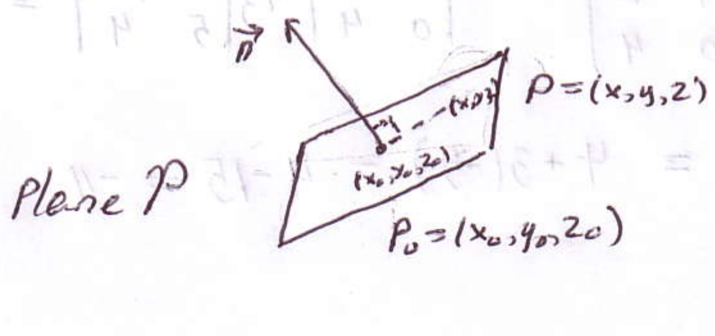
This motivates the following definition.

Def: The equation of the plane $\mathcal{P}$ through (x_0, y_0, z_0) that has a normal vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ is

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \, ;$$

that is, $(x,y,z) \in \mathcal{P}$ iff

$$Ax + By + Cz + D = 0 \quad \text{where } D = -Ax_0 - By_0 - Cz_0$$



$P \in \mathcal{P}$ iff

$\vec{n} \cdot (\overrightarrow{P - P_0}) = 0$

The four numbers A, B, C and D are not determined uniquely by the plane $\mathcal{P}$. To see this, note that (x,y,z) satisfies the equation $Ax+By+Cz+D=0$ iff it also satisfies the relation

$$(\lambda A)x + (\lambda B)y + (\lambda C)z + (\lambda D) = 0$$

for any constant $\lambda \neq 0$. Furthermore, if A, B, C, D and A', B', C', D' determine the same plane $\mathcal{P}$, then $A=\lambda A'$, $B=\lambda B'$, $C=\lambda C'$, $D=\lambda D'$ for a scalar λ. Consequently, A, B, C, D are determined by $\mathcal{P}$ up to a scalar multiple (why?).

Ex. Determine an equation for the plane that is perpendicular to the vector $\vec{i}+\vec{j}+\vec{k}$ and contains the point $(1,0,0)$.

Solution:

$\vec{n} = (1,1,1)$. Hence (x,y,z) is in $\mathcal{P}$ iff $\vec{n}\cdot(x-1, y-0, z-0) = 0$

so the equation is $(1,1,1)\cdot(x-1,y,z) = x-1+y+z = 0$

The equation of the plane can also be written as $x+y+z=1$.

Ex. Find an equation for the plane containing the three points $(1,1,1)$, $(2,0,0)$, and $(1,1,0)$.

Solution:

Pick one of the points to be P_0, say $(2,0,0)$. Then the vectors $\vec{a} = (1,1,1)-(2,0,0) = (-1,1,1)$ and $\vec{b} = (1,1,0)-(2,0,0) = (-1,1,0)$ lie on the plane.

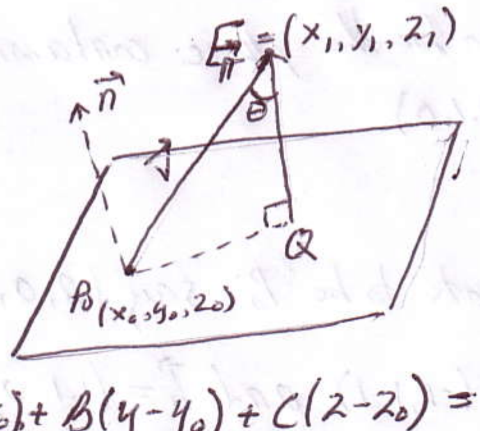
Hence the normal $\vec{n}$ to the plane must be perpendicular to both vectors $\vec{a}$ and $\vec{b}$. That is $\vec{n}\cdot\vec{a}=0$ and $\vec{n}\cdot\vec{b}=0$.
By the work done before we can take $\vec{n}$ to be the vector

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -\vec{i}-\vec{j} = (-1,-1,0) = \vec{n}$$

Our equation is therefore $\vec{n}\cdot(x-2, y-0, z-0)=0$ or
$(-1,-1,0)\cdot(x-2,y,z) = -(x-2)-y=0$ which we can also write as $x+y=2$

Two planes are called parallel when their normal vectors are parallel.
Thus the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ are parallel when $n_1 = A_1\vec{i}+B_1\vec{j}+C_2\vec{k}$ and $n_2 = A_2\vec{i}+B_2\vec{j}+C_2\vec{k}$ are parallel; that is, $\vec{n}_1 = \sigma\vec{n}_2$ for a constant σ. For example, the planes $x-2y+z=0$ and $-2x+4y-2z=10$ are parallel, while $x-2y+z=0$ and $2x-2y+z=10$ are not parallel.

<u>Distance: Point to Plane.</u>



Consider the problem of finding the distance from a point (x_1,y_1,z_1) to the plane with equation $A(x-x_0)+B(y-y_0)+C(z-z_0) = Ax+By+Cz+D=0$
Notice that the vector $\vec{QE}$ is parallel to the normal vector $\vec{n}$.

Thus, if $\vec{v} = \vec{P_0E} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$, $\vec{\omega} = \vec{QE}$ and θ is an angle between them, then $d(E, Q)$, the distance from E to the plane (which equals to distance from E to Q), is determined by $d(E,Q) = \|\vec{\omega}\| = \|P_{\vec{\omega}}(\vec{v})\| = \|P_{\vec{n}}(\vec{v})\| = \left\|\left(\frac{\vec{v}\cdot\vec{n}}{\|\vec{n}\|^2}\right)\vec{n}\right\| =$

$$= \frac{|\vec{v}\cdot\vec{n}|}{\|\vec{n}\|^2}\|\vec{n}\| = \frac{|\vec{v}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|(x_1-x_0, y_1-y_0, z_1-z_0)\cdot(A,B,C)|}{\sqrt{A^2+B^2+C^2}} =$$

$$= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}$$

Ex. Find the distance from $E = (2, 0, -1)$ to the plane $3x - 2y + 8z + 1 = 0$

Solution:

$\vec{n} = (3, -2, 8)$ so the distance is $\frac{|3\cdot 2 + (-2)\cdot 0 + 8(-1) + 1|}{\sqrt{3^2 + (-2)^2 + 8^2}} = \frac{|-1|}{\sqrt{77}}$

$= \frac{1}{\sqrt{77}}.$